# Physics 513, Quantum Field Theory

Homework 9

Due Tuesday, 18th November 2003

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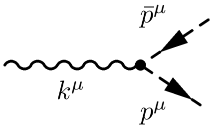
**The Decay of Vector into Two Scalars**

We are to compute the decay rate of unpolarized vector particles of mass $M$ into two scalars of mass $m$. We should calculate the decay rate in the rest frame.

Defining $\tilde{p}^\mu = (\bar{p} - p)^\mu$, the amplitude for the decay diagram is given by




$$= i\mathcal{M} = \epsilon_\mu i f \tilde{p}^\mu.$$

It is quite straightforward to calculate the spin-averaged square of the amplitude,

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{3}\sum_{\text{spin}} \epsilon_\mu i f \tilde{p}^\mu \epsilon_\nu^* (-i) f \tilde{p}^\nu, \\
&= \frac{f^2}{3}\left(\frac{k_\mu k_\nu}{M^2} - g_{\mu\nu}\right)\tilde{p}^\mu \tilde{p}^\nu, \\
&= \frac{f^2}{3}\left(\frac{(k_\mu \tilde{p}^\mu)^2}{M^2} - \tilde{p}^2\right).
\end{aligned}$$

Now, because we are computing this in the rest frame where $k_\mu = (M, 0)$ and $\tilde{p}^\mu = (0, -2|\vec{p}|)$, $k_\mu \tilde{p}^\mu = 0$. Similarly, we know that $\tilde{p}^2 = 4|\vec{p}|^2$. Therefore,

$$\overline{|\mathcal{M}|^2} = \frac{4f^2|\vec{p}|^2}{3}.$$

Note that $|\vec{p}| = E^2 - m^2 = \left(\frac{M^2}{4} - m^2\right)^{1/2}$. Using this and the equation for the decay rate found in Peskin and Schroeder,

$$\begin{aligned}
\Gamma &= \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{p}|}{M}|\mathcal{M}|^2, \\
&= \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{p}|}{M}\frac{4f^2|\vec{p}|^2}{3}, \\
&= \frac{f^2}{24\pi^2 M^2}\int d\Omega |\vec{p}|^3,
\end{aligned}$$

$$\therefore \Gamma = \frac{f^2\left(\frac{M^2}{4} - m^2\right)^{3/2}}{6\pi M^2}.$$

**Mott's Formula**

We are to generalize problem 2 of Homework 8 in the relativistic case. We computed then the general amplitude to be

$$\mathcal{M} = \frac{-ie^2 Z}{(p_f - p)^2}\bar{u}^{s'}(p_f)\gamma^0 u^s(p).$$

To compute the spin averaged amplitude, it will be helpful to recall our earlier kinematic result that $(p_f - p)^4 = 16|\vec{p}|^4 \sin^4\theta/2$. Let us now compute the amplitude squared in the spin-averaged case.

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{2}\frac{Z^2 e^4}{(p_f - p)^4}\sum_{\text{spin}} \bar{u}^s(p)\gamma^0 u^{s'}(p_f)\bar{u}^{s'}(p_f)\gamma^0 u^s(p), \\
&= \frac{Z^2 e^4}{32|\vec{p}|^4 \sin^4\theta/2}\text{Tr}\left(\gamma^0(\not{p}_f + m)\gamma^0(\not{p} + m)\right).
\end{aligned}$$

It will be helpful to break up the trace into its four additive pieces.

$$\text{Tr}\left(\gamma^0(\not{p}_f + m)\gamma^0(\not{p} + m)\right) = \text{Tr}\left(\gamma^0\not{p}_f\gamma^0\not{p}\right) + \text{Tr}\left(\gamma^0 m\gamma^0\not{p}\right) + \text{Tr}\left(\gamma^0\not{p}_f\gamma^0 m\right) + \text{Tr}\left(\gamma^0 m\gamma^0 m\right).$$

It should be clear that the two middle terms are both zero because there is an odd number of $\gamma$'s. The last term is nearly trivial, $\text{Tr}\left(\gamma^0 m\gamma^0 m\right) = 4m^2$. Let us now work on the first term.

$$\begin{aligned}
\text{Tr}\left(\gamma^0 \not{p}_f \gamma^0 \not{p}\right) &= p_{f_\mu} p_\nu \text{Tr}\left(\gamma^0\gamma^\mu\gamma^0\gamma^\nu\right), \\
&= 4 p_{f_\mu} p_\nu \left(g^{0\mu}g^{0\nu} - g^{00}g^{\mu\nu} + g^{0\nu}g^{\mu 0}\right), \\
&= 4\left(2E^2 - p_{f_\mu}p^\mu\right), \\
&= 4\left(2E^2 - E^2 + \vec{p_f}\vec{p}\right), \\
&= 4\left(E^2 + |\vec{p}|^2\cos\theta\right).
\end{aligned}$$

Using these results, we have that

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[E^2 + |\vec{p}|^2\cos\theta + m^2\right], \\
&= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[2E^2 - |\vec{p}|^2(1-\cos\theta)\right], \\
&= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[2E^2 - 2|\vec{p}|^2\sin^2\theta/2\right], \\
&= \frac{Z^2e^4E^2}{4|\vec{p}|^4\sin^4\theta/2}\left[1 - \left(\frac{|\vec{p}|}{E}\right)^2\sin^2\theta/2\right], \\
&= \frac{Z^2e^4}{4\beta^2|\vec{p}|^2\sin^4\theta/2}\left[1 - \beta^2\sin^2\theta/2\right].
\end{aligned}$$

In the last two lines we have used the fact that $\vec{p}/E = \beta$. Now, we showed in Homework 8 that

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2}.$$

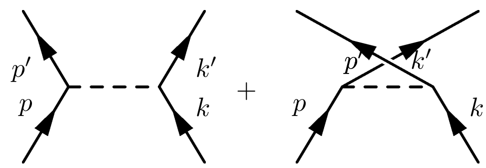
Using the fine structure constant to simplify notation, where $\alpha^2 = \frac{e^4}{16\pi^2}$, it is clear that

$$\therefore \frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4\beta^2|\vec{p}|^2\sin^4\theta/2}\left[1 - \beta^2\sin^2\theta/2\right].$$

**Helicity Amplitudes in Yukawa Theory**

We are to consider the amplitude given by,

$$i\mathcal{M} =$$




$$= (-ig^2)\left(\bar{u}(p')u(p)\frac{1}{(p'-p)^2-m_\phi^2}\bar{u}(k')u(k) - \bar{u}(p')u(k)\frac{1}{(p'-k)^2-m_\phi^2}\bar{u}(k')u(p)\right).$$

**a)** We are to derive the selection rules for helicity for this theory.

We can best understand the selection rules by requiring that one of the spinors is in a projection. To bring the projection operator to the neighboring spinor (in either diagram and starting from any outside term) requires that the projection anticommutes through a $\gamma^0$. Therefore, the interaction *must* flip the spins. Exempli Gratia, $\bar{u}\frac{1+\gamma^5}{2}u_R = u^\dagger\gamma^0\frac{1+\gamma^5}{2}u_R = \bar{u}_L u_R$.

**b)** Given these selection rules, what are the non-vanishing amplitudes? These are the only possible terms that involve both incoming states flipping their spin in the outgoing states. So, the nonzero amplitudes are $\mathcal{M}_{LL;RR}, \mathcal{M}_{RR;LL}, \mathcal{M}_{LR;RL}, \mathcal{M}_{RL;LR}, \mathcal{M}_{RL;RL}, \mathcal{M}_{LR;LR}$.

**c)** We are to use problem 5 of Homework 5 to compute the explicit form of the two-spinors. We should use this to find the eigenvectors $u_\lambda(p)$ at very high energies. This is a relatively straight forward calculation. We derived quite some time ago that in the high energy limit for general spinors. Using the helicity basis derived in Homework 5, we see that

$$u_R = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \qquad \text{and} \qquad u_L = \sqrt{2E}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ 0 \\ 0 \end{pmatrix}.$$

**d)** Now we should rederive the selection rules from part (a). This is relatively straight forward. Let us compute directly the $\bar{u}_R$ and $\bar{u}_L$ These two are simply,

$$\bar{u}_R = \left(-\sqrt{2E}e^{i\phi}\sin\theta/2, \sqrt{2E}\cos\theta/2, 0, 0\right) \qquad \text{and} \qquad \bar{u}_L = \left(0, 0, \sqrt{2E}\cos\theta/2, \sqrt{2E}e^{-i\phi}\sin\theta/2\right).$$

It should be clear that in this limit, $\bar{u}_R u_R = 0$ because the have opposite zeros. Therefore, we may again conclude that the only inner products that do not vanish are those which flip the spin at the vertex. This is the same relationship seen intuitively in part (a).

**e)** We must now compute the nonvanishing inner produces of the eigenvectors that we mentioned above. Let us compute each in turn directly.

$$\begin{aligned}
\bar{u}_R(p')u_L(p) &= -2Ee^{i\phi}\sin\theta/2;\\
\bar{u}_L(p')u_R(p) &= 2Ee^{-i\phi}\sin\theta/2;\\
\bar{u}_R(k')u_L(p) &= -2Ee^{i\phi}\cos\theta/2;\\
\bar{u}_L(k')u_R(p) &= 2Ee^{-i\phi}\cos\theta/2;\\
\bar{u}_R(p')u_L(k) &= 2Ee^{i\phi}\cos\theta/2;\\
\bar{u}_L(p')u_R(k) &= -2Ee^{-i\phi}\cos\theta/2;\\
\bar{u}_R(k')u_L(k) &= 2Ee^{i\phi}\sin\theta/2;\\
\bar{u}_L(k')u_R(k) &= -2Ee^{-i\phi}\sin\theta/2.
\end{aligned}$$

**f)** Let us compute the amplitudes $\mathcal{M}_{RR;LL}$ and $\mathcal{M}_{LR;LR}$ in the limit of very high energy. We use the limit to reduce $|\vec{p}|^2$-like terms to $E^2$. These are directly computed to be

$$\begin{aligned}
\mathcal{M}_{RR;LL} &= -g^2\left((-2Ee^{i\phi}\sin\theta/2)\frac{1}{4E^2\sin^2\theta/2}2Ee^{i\phi}\sin\theta/2 - 2Ee^{i\phi}\cos\theta/2\frac{1}{-4E^2\cos^2\theta/2}(-2Ee^{i\phi}\cos\theta/2)\right),\\
&= g^2(e^{i\phi}+e^{i\phi}),
\end{aligned}$$

$$\therefore \mathcal{M}_{RR;LL} = 2g^2e^{i\phi}.$$

By a similar calculation,

$$\mathcal{M}_{LR;LR} = -g^2\left(2Ee^{-i\phi}\cos\theta/2\frac{1}{-4E^2\cos^2\theta/2}(-2Ee^{i\phi}\cos\theta/2)\right),$$

$$\therefore \mathcal{M}_{LR;LR} = -g^2.$$

**g)** Let us determine the spin averaged amplitude squared. The contributions are very similar to the two above (in fact, the amplitudes are identical so we just multiply). We see

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\left(2(2g^2)^2 + 4g^4\right),$$

$$\therefore \overline{|\mathcal{M}|^2} = 3g^4.$$